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On the Convergence of Cardinal Logarithmic Splines

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1. INTRODUCTION

Let $\mathbb{R}_+^\times = \{x \in \mathbb{R} \mid x > 0\}$ denote the open positive real half-line. Throughout this paper let an arbitrary real number $h_0 > 1$ (the initial step width) and the function

$$f_0 : \mathbb{R}_+^\times \ni x \rightsquigarrow (\log x)/\log h_0 \in \mathbb{R} \tag{1}$$

be fixed. In a recent paper Newman and Schoenberg [8] have investigated the convergence behavior of the cardinal logarithmic splines $(S_m)_{m\geqslant 1}$ on $\mathbb{R}_+^\times$ that interpolate the function f_0 at the geometric sequence $(h_0^n)_{n\in\mathbb{Z}}$ of nodes; for the exact definitions of these notions see Section 2. The technique used by these autors is based on a rather tricky exploitation of a Fourier series expansion. The purpose of the present article however is to study the convergence properties of the sequence $(S_m)_{m\geqslant 1}$ by means of a method that is of quite a different nature. This approach is adapted to the limiting form of the sequence $(S_m)_{m\geqslant 1}$ as the degree m tends to infinity and relies on an appropriate second-order refinement of Karamata's Abel–Tauber theorem for the one sided Laplace transform.

2. CARDINAL LOGARITHMIC SPLINES

Let $m \in \mathbb{N}^\times$ be an arbitrary natural number and $S_m \in \mathscr{C}^{m-1}(\mathbb{R}_+^\times)$ a real-valued function such that the restriction of S_m to every compact interval $[h_0^n, h_0^{n+1}]$ ($n \in \mathbb{Z}$) of the half-line $\mathbb{R}_+^\times$ is a polynomial function of degree $\leqslant m$. Following Newman and Schoenberg [8] the function S_m is called a *cardinal logarithmic spline of degree m* if it satisfies the following two conditions:

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(i) S_m interpolates the function f_0 at the geometric sequence $(h_0{}^n)_{n\in\mathbb{Z}}$ of nodes, i.e., S_m satisfies

$$S_m(h_0{}^n) = f_0(h_0{}^n) = n \qquad \text{(IP)}$$

for each number $n \in \mathbb{Z}$ ("*interpolation property*").

(ii) S_m satisfies the inhomogeneous linear geometric difference equation of the first order

$$S_m(h_0x) - S_m(x) = 1 \qquad x \in \mathbb{R}_+{}^\times) \qquad \text{(DE)}$$

("*difference equation*"; see Meschkowski [7, Kap. XV]).

The conditions (IP), (DE) are sufficient to determine the function S_m uniquely. Moreover, the cardinal logarithmic spline of degree $m \geqslant 1$ admits the explicit representation

$$S_m(x) = \sum_{n\in\mathbb{Z}} ((1 - h_0^{-n})_+^m - (1 - xh_0^{-n})_+^m) \qquad (x \in \mathbb{R}_+{}^\times) \qquad (2)$$

where for any function $f\colon \mathbb{R}_+{}^\times \to \mathbb{R}$ the symbol $f_+ = \sup\{f, 0\}$ stands for the positive part of f. We obtain by (2) the relation

$$\lim_{m\to\infty} S_m(x) = \lim_{m\to\infty} \sum_{n\in\mathbb{Z}} (e^{-h_0^{(f_0(m)+n)}} - e^{-xh_0^{(f_0(m)+n)}}), \qquad (3)$$

the limit being uniform for all points $x \in \mathbb{R}_+{}^\times$. For each number $y \in \mathbb{R}$ let $[y] = \sup\{n \in \mathbb{Z} \mid n \leqslant y\}$ be the greatest integer $\leqslant y$. Consider the saltus function

$$g\colon \mathbb{R}_+{}^\times \ni t \rightsquigarrow [f_0(t)] + 1 \in \mathbb{R}. \qquad (4)$$

Observe that g is nondecreasing and continuous on the right and that it satisfies the geometric difference equation (DE)

$$g(th_0) - g(t) = 1 \qquad (t \in \mathbb{R}_+{}^\times). \qquad (5)$$

It follows that there exists a unique positive Radon measure μ_g on $\mathbb{R}_+{}^\times$ such that

$$\mu_g(]a, b]) = g(b) - g(a) \qquad (6)$$

holds for every half-open interval $]a, b] \subset \mathbb{R}_+{}^\times$, the Lebesgue–Stieltjes measure on $\mathbb{R}_+^\times$ defined by g. Then, for any number $\epsilon > 0$ the Laplace–Stieltjes integrals

$$\mathscr{L}_\epsilon \mu_g(t) = \int_\epsilon^{+\infty} e^{-ts}\, d\mu_g(s) \qquad (t \in \mathbb{R}_+^\times) \tag{7}$$

exist and we obtain the relation

$$\lim_{\epsilon \to 0+} (\mathscr{L}_\epsilon \mu_g(th_0) - \mathscr{L}_\epsilon \mu_g(t)) = -1 \qquad (t \in \mathbb{R}_+^\times). \tag{8}$$

It follows from (3), definitions (4), (6), and from (8) that *the approximation property of the cardinal logarithmic splines* $(S_m)_{m \geqslant 1}$ *on the helf-line* $\mathbb{R}_+^\times$ *holds at* $x \in \mathbb{R}_+^\times$, *i.e.*,

$$\lim_{m \to \infty} S_m(x) = f_0(x) \qquad (x \in \mathbb{R}_+^\times) \tag{APx}$$

if and only if the relation

$$\lim_{m \to \infty} \left(\lim_{\epsilon \to 0+} \frac{\mathscr{L}_\epsilon \mu_g(mx) - \mathscr{L}_\epsilon \mu_g(m)}{\mathscr{L}_\epsilon \mu_g(mh_0) - \mathscr{L}_\epsilon \mu_g(m)} \right) = f_0(x) \qquad (x \in \mathbb{R}_+^\times) \tag{9}$$

is satisfied.

3. Application of the Abel–Tauber Theorem

Clearly, (IP) implies (APx) for any point $x \in \mathbb{R}_+^\times$ that belongs to the sequence $(h_0^n)_{n \in \mathbb{Z}}$ of interpolation nodes. To prove the converse of this assertion, i.e., to prove that (APx) holds only if $x \in \mathbb{R}_+^\times$ reveals to be one of the nodes $(h_0^n)_{n \in \mathbb{Z}}$ we shall exploit the limiting relation (9).

Let any (linear) transformation between function spaces be given. It is well known that each result that provides information concerning the asympotic behavior of the transformed function by means of the asymptotic behavior of the original function is called an *Abelian* theorem. Conversely, any theorem that gives information concerning the asymptotic behavior by means of a conclusion in the opposite direction (from the transformed function to the given one) is called a *Tauberian* theorem. See, for instance, Beekmann and Zeller [10] and in particular, for the case of the Laplace transform, Doetsch [4, Part V]. For our purposes the following Abel–Tauber theorem essentially due to de Haan [1–3] is of importance. It represents a suitable refinement of Karamata's theorem for one-sided Laplace transforms (Hardy [6, Chap. VII]; Widder [9, Chap. 8]).

THEOREM 1. *Let $x \in \mathbb{R}_+^\times$ be any fixed point. The limiting relation* (9) *holds if and only if the condition*

$$\lim_{t\to 0+} \frac{g(tx) - g(t)}{g(th_0) - g(t)} = f_0(x) \tag{10}$$

is satisfied by the saltus function g.

Proof (Sketch). For each number $\epsilon > 0$ define the mapping

$$\begin{aligned} g_\epsilon : \mathbb{R}_+^\times \ni t &\rightsquigarrow 0 && \text{if} \quad t \in]0, \epsilon[, \\ &\rightsquigarrow g(t) - g(\epsilon) && \text{if} \quad t \in [\epsilon, +\infty[. \end{aligned}$$

Then $(g_\epsilon)_{\epsilon>0}$ is a family of nondecreasing saltus functions $\geqslant 0$ on $\mathbb{R}_+^\times$ that are continuous on the right and satisfy $g_\epsilon(0+) = 0$. Furthermore, we have

$$\mathscr{L}_\epsilon \mu_g = \mathscr{L}_{0+} \mu_{g_\epsilon} \qquad (\epsilon > 0). \tag{12}$$

In view of the identity

$$\lim_{t\to 0+} \frac{g(tx) - g(t)}{g(th_0) - g(t)} = \lim_{t\to 0+} \left(\lim_{\epsilon\to 0+} \frac{g_\epsilon(tx) - g_\epsilon(t)}{g_\epsilon(th_0) - g_\epsilon(t)} \right) \qquad (x \in \mathbb{R}_+^\times) \tag{13}$$

an application of Theorem 1.4.3 in de Haan's tract [1] shows that the condition (10) is satisfied at the point $x \in \mathbb{R}_+^\times$ if and only if the functions

$$G_\epsilon : \mathbb{R}_+^\times \ni t \rightsquigarrow g_\epsilon(t) - \frac{1}{t} \int_0^t g_\epsilon(s)\, ds \in \mathbb{R} \qquad (\epsilon > 0) \tag{14}$$

have the property $\lim_{t\to 0+}(\lim_{\epsilon\to 0+} G_\epsilon(tx) - G_\epsilon(t)) = 0$, i.e., if and only if the family $(G_\epsilon)_{\epsilon>0}$ is in the limit $\epsilon \to 0+$ "slowly varying" at zero on the right. Analogous arguments as used in [3] and [1]—we do not repeat all of the details—show the equivalence of (10) and the relation

$$\lim_{m\to\infty} \left(\lim_{\epsilon\to 0+} \frac{\mathscr{L}_{0+}\mu_{g_\epsilon}(mx) - \mathscr{L}_{0+}\mu_{g_\epsilon}(m)}{\mathscr{L}_{0+}\mu_{g_\epsilon}(mh_0) - \mathscr{L}_{0+}\mu_{g_\epsilon}(m)} \right) = f_0(x) \qquad (x \in \mathbb{R}_+^\times). \tag{15}$$

In view of (12) the assertion follows.

Let us switch back to our problem. In view of (5), (4), and (1) we obtain the equivalence of (10) with the identity

$$\lim_{t\to 0+} ([f_0(t) + f_0(x)] - [f_0(t)]) = f_0(x), \tag{16}$$

i.e., with $[f_0(x)] = f_0(x)$ for any $x \in \mathbb{R}_+^\times$. Thus, Theorem 1 supra implies immediately the following (unexpected) convergence theorem due to Newman and Schoenberg [8].

THEOREM 2. *The sequence* $(S_m)_{m \geqslant 1}$ *of cardinal logarithmic splines on* $\mathbb{R}_+^\times$ *satisfies* (APx) *at the point* $x \in \mathbb{R}_+^\times$ *if and only if* x *belongs to the sequence* $(h_0^n)_{n \in \mathbb{Z}}$ *of interpolation nodes.*

For the limiting behavior of cardinal spline functions on $\mathbb{R}$ that satisfy the homogeneous linear difference equation of the first order $S(x + 1) - hS(x) = 0$, $x \in \mathbb{R}$, the reader is referred to the article by Greville *et al.* [5] and the literature cited therein.

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REFERENCES

1. L. DE HAAN, "On Regular Variation and Its Application to the Weak Convergence of Sample Extremes," Mathematical Centre Tracts Vol. 32, Mathematisch Centrum, Amsterdam, 1970.
2. L. DE HAAN, A form of regular variation and its application to the domain of attraction of the double exponential distribution. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **17** (1971), 241–258.
3. L. DE HAAN, An Abel–Tauber theorem for Laplace transforms. *J. London Math. Soc.* **13** (1976), 537–542.
4. G. DOETSCH, "Handbuch der Laplace-Transformation," Vol. I. Birkhäuser Verlag, Basel/Stuttgart, 1971.
5. T. N. E. GREVILLE, I. J. SCHOENBERG, AND A. SHARMA, The spline interpolation of sequences satisfying a linear recurrence relation. *J. Approximation Theory* **17** (1976), 200–221.
6. G. H. HARDY, "Divergent Series," Oxford Univ. Press (Clarendon), London, 1973.
7. H. MESCHKOWSKI, "Differenzengleichungen," Vandenhoeck and Ruprecht, Göttingen, 1959.
8. D. J. NEWMAN AND I. J. SCHOENBERG, Splines and the logarithmic function. *Pacific J. Math.* **61** (1975), 241–258.
9. D. V. WIDDER, "An Introduction to Transform Theory," Academic Press, New York/London, 1971.
10. K. ZELLER AND W. BEEKMANN, "Theorie der Limitierungsverfahren," 2nd ed., Springer Verlag, Berlin/Heidelberg/New York 1970.